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ON FIXED-DURATION DYNAMIC GAMES*

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Four methods of successive approximations are examined for a dynamic game's value function, which are used to construct the players' ε-optimal strategies. General piecewise-programmed strategies are used to prove the convergence of these methods. The paper's contents about the investigations in /1—9/ and generalize the results in /8,9/ to the case of a general dynamic system.

1. Let there be specified a time interval $[t_0, T]$, a state set X, sets U and V, the first (second) player's control set D_1 (D_2) whose elements are mappings of the time interval $[t_0, T]$ into U (V), and the state function

$$\varkappa : [t_0, T] \times [t_0, T] \times X \times D_1 \times D_2 \to X$$

The quintuple $\Sigma = ([t_0, T], X, D_1, D_2, \varkappa)$ is called a general dynamic system if the following conditions are fulfilled:

1) sets X, D_1, D_2 are nonempty;

2) if $u_1, u_2 \in D_1$, $v_1, v_2 \in D_2$ and $t_0 \leqslant t_1 < t_2 < t_3 \leqslant T$, then we can find $u_3 \in D_1$, $v_3 \in D_2$ such that

$$u_3(t) = \begin{cases} u_1(t), & t_1 \leqslant t < t_2, \\ u_2(t), & t_2 \leqslant t < t_3, \end{cases} \quad v_3(t) = \begin{cases} v_1(t), & t_1 \leqslant t < t_2 \\ v_2(t), & t_2 \leqslant t < t_3 \end{cases}$$

3) the function $x = \varkappa(t, \tau, x_*, u, v)$ is defined for all $t \geqslant \tau$ and is not necessarily defined for all $t < \tau$;

4) the equality $\varkappa(t, t, x, u, v) = x$ is fulfilled for any $t_0 \leqslant t \leqslant T$, $x \in X$, $u \in D_1$, $v \in D_2$;

5) the equality

$$\varkappa(t_3, t_1, x, u, v) = \varkappa(t_3, t_2, \varkappa(t_2, t_1, x, u, v), u, v)$$

is fulfilled for any $t_0 \leqslant t_1 < t_2 < t_3 \leqslant T$ and any $x \in X$, $u \in D_1$, $v \in D_2$;

6) if $u_1, u_2 \in D_1$, $v_1, v_2 \in D_2$ and $u_1(t) = u_2(t)$, $v_1(t) = v_2(t)$ when $t_0 \leqslant t_1 \leqslant t < t_2 \leqslant T$, then for any $x \in X$ we have

$$\varkappa(t_2, t_1, x, u_1, v_1) = \varkappa(t_2, t_1, x, u_2, v_2)$$

An element $x(t) = \varkappa(t, t_*, x_*, u, v)$ of set X is called a state of system Σ at the instant t, while the corresponding mapping $x(\cdot)\cdot [t_*, T] \to X$ is called a trajectory of system Σ if at instant t_* the system is found to be in state x_* and the controls u and v are acting on it. By $D_k[t_1, t_2)$ we denote the set of all restrictions of the k-th player's controls to the interval $[t_1, t_2)$, $k = 1, 2$.

2. We examine dynamic games $\Gamma(t_*, x_*)$ described by system Σ, which the first player's payoff is

$$I(u, v, t_*, x_*) = H(\varkappa(T, t_*, x_*, u, v))$$

where $H: X \to R^1$. We assume that the first player maximizes this functional, while the second minimizes it. We also assume that the players use piecewise-programmed strategies /3—7/. We consider the following four sequences, the first two of which are generalizations of the sequences in /8,9/:

$$V_1^+(t, x) = W_1^+(t, x) = \inf_{v \in D_2} \sup_{u \in D_1} H(\varkappa|_{\tau=T})$$

$$V_1^-(t, x) = W_1^-(t, x) = \sup_{u \in D_1} \inf_{v \in D_2} H(\varkappa|_{\tau=T})$$

$$V_n^+(t, x) = \inf_{\tau \in [t, T]} \inf_{v \in D_2} \sup_{u \in D_1} V_{n-1}^+(\tau, \varkappa)$$

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$$
\begin{aligned}
&V_n^-(t,x) = \sup_{\tau\in[t,T]}\ \sup_{u\in D_1}\ \inf_{v\in D_2} V_{n-1}^-(\tau,\varkappa)\\
&W_n^+(t,x) = \inf_{v\in D_2}\ \sup_{u\in D_1}\ \inf_{\tau\in[t,T]} W_{n-1}^+(\tau,\varkappa)\\
&W_n^-(t,x) = \sup_{u\in D_1}\ \inf_{v\in D_2}\ \sup_{\tau\in[t,T]} W_{n-1}^-(\tau,\varkappa)\\
&t_0\leqslant t\leqslant T,\quad x\in X,\quad n=2,3,\ldots;\quad \varkappa=\varkappa(\tau,t,x,u,v)
\end{aligned}
$$

We assume that the dynamic system Σ satisfies the condition:

7) for any $t_0\leqslant t_1<t_2\leqslant T$, $x_1, x_2\in X$ there exist controls $u_*=u(t_1,t_2,x_1,x_2)\in D_1[t_1,t_2)$ and $v_*=v(t_1,t_2,x_1,x_2)\in D_2[t_1,t_2)$ such that

$$
\{d[\varkappa(t,t_1,x_1,u_*,v),\varkappa(t,t_1,x_2,u,v_*)]\}^m\leqslant\{d[x_1,x_2]\}^m\exp\beta(t_2-t_1)+\gamma(t_2-t_1)(t_2-t_1) \tag{2.1}
$$
$$
t_1\leqslant t\leqslant t_2,\quad \lim_{\delta\to 0}\gamma(\delta)=0,\quad m,\ \beta>0
$$

for all $u\in D_1$, $v\in D_2$, where d is some metric on state set X.

Theorem 1. If the dynamic system Σ satisfies condition 7) and the function H is uniformly continuous, then for any number $\varepsilon>0$ a pair of ε-optimal strategies exists in game $\Gamma(t_*,x_*)$ and the condition

$$
V(t,x)=\lim V_n^+(t,x)=\lim V_n^-(t,x)=\lim W_n^+(t,x)=\lim W_n^-(t,x),\quad n\to\infty \tag{2.2}
$$

is fulfilled for the value function $V(t_*,x_*)=\operatorname{val}\Gamma(t_*,x_*)$.

3. For dynamic systems satisfying the condition

8) $U\subset D_1$, $V\subset D_2$

we examine the quantities

$$
\begin{aligned}
&V_{c1}^+(t,x)=W_{c1}^+(t,x)=\inf_{v\in V}\ \sup_{u\in D_1} H(\varkappa|_{\tau=T})\\
&V_{c1}^-(t,x)=W_{c1}^-(t,x)=\sup_{u\in U}\ \inf_{v\in D_2} H(\varkappa|_{\tau=T})\\
&V_{cn}^+(t,x)=\inf_{\tau\in[t,T]}\ \inf_{v\in V}\ \sup_{u\in D_1} V_{c(n-1)}^+(\tau,\varkappa)\\
&V_{cn}^-(t,x)=\sup_{\tau\in[t,T]}\ \sup_{u\in U}\ \inf_{v\in D_2} V_{c(n-1)}^-(\tau,\varkappa)\\
&W_{cn}^+(t,x)=\inf_{v\in V}\ \sup_{u\in D_1}\ \inf_{\tau\in[t,T]} W_{c(n-1)}^+(\tau,\varkappa)\\
&W_{cn}^-(t,x)=\sup_{u\in U}\ \inf_{v\in D_2}\ \sup_{\tau\in[t,T]} W_{c(n-1)}^-(\tau,\varkappa)\\
&t_0\leqslant t\leqslant T,\quad x\in X,\quad n=2,3,\ldots;\quad \varkappa=\varkappa(\tau,t,x,u,v)
\end{aligned}
$$

Let the following condition be fulfilled:

9) for all $t_0\leqslant t_1\leqslant T$, $x_1, x_2\in X$ there exist controls $u_*=u(t_1,x_1,x_2)\in U$, $v_*=v(t_1,x_1,x_2)\in V$ such that (2.1) is fulfilled for all $t_1\leqslant t=t_2\leqslant T$, $u\in D_1$, $v\in D_2$.

Theorem 2. If conditions 8) and 9) are fulfilled for a dynamic system Σ and function H is uniformly continuous, then game Γ_* has a value and the equalities

$$
\begin{aligned}
V_*=\operatorname{val}\Gamma_*&=\lim V_{n*}^+=\lim V_{n*}^-=\lim W_{n*}^+=\lim W_{n*}^+=\\
&\lim V_{cn*}^+=\lim V_{cn*}^-=\lim W_{cn*}^+=\lim W_{cn*}^-
\end{aligned} \tag{3.1}
$$

are valid.

Here and subsequently the asterisk signifies that t_* and x_* are the arguments of the function in question; the limit is taken as $n\to\infty$.

4. Let us prove Theorems 1 and 2. For any finite partitioning

$$
\Delta=\{t_*=t_0^\Delta<t_1^\Delta<t_2^\Delta<\ldots<t_{n(\Delta)}^\Delta=T\}
$$

the dynamic games with discrimination Γ_*^Δ, $\Gamma_{\Delta*}$ have the values /6,7,10/

$$
V_*^\Delta=\operatorname{val}\Gamma_*^\Delta=\inf_{u_1}\ \sup_{v_1}\ldots\inf_{v_{n(\Delta)}}\ \sup_{u_{n(\Delta)}} H(\varkappa) \tag{4.1}
$$
$$
V_{\Delta*}=\operatorname{val}\Gamma_{\Delta*}=\sup_{u_1}\ \inf_{v_1}\ldots\sup_{u_{n(\Delta)}}\ \inf_{v_{n(\Delta)}} H(\varkappa) \tag{4.2}
$$
$$
\begin{aligned}
&u_k\in D_1[t_{k-1}^\Delta,t_k^\Delta),\quad v_k\in D_2[t_{k-1}^\Delta,t_k^\Delta),\quad k=1,2,\ldots,n(\Delta)\\
&\varkappa=\varkappa(T,t_*,x_*;u_1,\ldots,u_{n(\Delta)};v_1,\ldots,v_{n(\Delta)})
\end{aligned}
$$

Let the quantity V_{c*}^{Δ} be defined by formula (4.1) wherein

$$u_k \in D_1 [t_{k-1}^{\Delta}, t_*^{\Delta}), \quad v_k \in V, \quad k = 1, 2, \ldots, n(\Delta)$$

and let the quantity $V_{\Delta*}^{c}$ be defined by formula (4.2) wherein

$$u_k \in U, \quad v_k \in D_2 [t_{k-1}^{\Delta}, t_k^{\Delta}), \quad k = 1, 2, \ldots, n(\Delta)$$

These quantities can be looked upon as the values of the dynamic games with discrimination Γ_{c*}^{Δ}, $\Gamma_{\Delta*}^{c}$ in which the player being discriminated uses piecewise-constant strategies /6/. From condition 7) and the uniform continuity of function H follows the equality (1,6,10/

$$V_* = \lim V_*^{\omega(n)} = \lim V_{\omega(n)*}, \quad n \to \infty \tag{4.3}$$
$$t_k^{\omega(n)} = t_* + k(T - t_*)/2^n, \quad k = 0, 1, 2, \ldots, 2^n$$

Analogously, from conditions 8) and 9) and the uniform continuity of H follows the equality

$$V_* = \lim V_{c*}^{\omega(n)} = \lim V_{\omega(n)}^{c} \tag{4.4}$$

The following statements are valid.

Lemma 1. $V_*^{\omega(n)} \geqslant V_{2^n*}^{+} \geqslant W_{n*}^{+} \geqslant W_{2^n*}^{-} \geqslant V_{2^n*}^{-} \geqslant V_{\omega(n)*}$, $n = 1, 2, \ldots$

Lemma 2. $V_{c*}^{\omega(n)} \geqslant V_{c2^n*}^{+} \geqslant W_{c2^n*}^{+} \geqslant W_{c2^n*}^{-} \geqslant V_{c2^n*}^{-} \geqslant V_{\omega(n)*}^{c}$, $n = 1, 2, \ldots$

All the sequencies in (2.2) and (3.1) are monotone. Consequently, Theorem 1 follows from (4.3) and Lemma 1, while Theorem 2 follows from (4.4) and Lemma 2.

Lemma 1. All the inequalities in this lemma, except the middle one

$$W_{m*}^{+} \geqslant W_{m*}^{-}, \; m = 1, 2, \ldots \tag{4.5}$$

follow from the definitions of the sequences being examined. To prove inequalities (4.5) we take advantage of the following concept.

Definition. The matrix

$$a_n = \begin{Vmatrix} \varepsilon_1^t & \varepsilon_2^t & \ldots & \varepsilon_n^t \\ \varphi_1^t & \varphi_2^t & \ldots & \varphi_n^t \end{Vmatrix}$$
$$\varepsilon_n^t = T - t, \; \varepsilon_k^t: X \times D_1 [t, T) \times D_2 [t, T) \to [0, T - t]$$
$$\varphi_k^t: X \to D_1 [t, T), \; t \in [t_0, T], \; k = 1, 2, \ldots, n$$

is called a general nth-order positional piecewise-programmed strategy for the first player in system Σ.

The general positional piecewise-programmed strategies for the second player are defined analogously. For each position $\{t_*, x_*\}$ any pair of general positional piecewise-programmed strategies

$$a = \begin{Vmatrix} \varepsilon_1^t & \varepsilon_2^t & \ldots & \varepsilon_n^t \\ \varphi_1^t & \varphi_2^t & \ldots & \varphi_n^t \end{Vmatrix}, \quad b = \begin{Vmatrix} \sigma_1^t & \sigma_2^t & \ldots & \sigma_m^t \\ \psi_1^t & \psi_2^t & \ldots & \psi_m^t \end{Vmatrix}$$

define a unique trajectory

$$x(t) = \varkappa(t, t_*, x_*, a, b) = \varkappa(t, t_*, x_*, u(a, b), v(a, b))$$

of system Σ in the following manner. At first the players choose the controls $u_1 = \varphi_1^{t_*}(x_*)$, $v_1 = \psi_1^{t_*}(x_*)$. For definiteness we assume that

$$\varepsilon_1^{t_*}(x_*, u_1, v_1) < \sigma_1^{t_*}(x_*, u_1, v_1)$$

Then at the instant $t_1^{(1)} = t_* + \varepsilon_1^{t_*}(x_*, u_1, v_1)$ the first player chooses the new control

$$u_2 = \varphi_2^{t_1^{(1)}}(x_1^{(1)}) \in D_1 [t_1^{(1)}, T), \; x_1^{(1)} = \varkappa(t_1^{(1)}, t_*, x_*, u_1, v_1)$$

For example, let

$$t_1^{(2)} = t_* + \sigma_1^{t_*}(x_*, u_1', u_2; v_1) < t_1^{(1)} + \varepsilon_2^{t_1^{(1)}}(x_1^{(1)}, u_2, v_1')$$

where u_1' is the restriction of u_1 to $[t_*, t_1^{(1)})$ and v_1' is the restriction of v_1 to $[t_1^{(1)}, T)$. Then at the instant $t_1^{(2)}$ the second player chooses the control

$$v_2 = \psi_2^{t_1^{(2)}}(x_1^{(2)}) \in D_2[t_1^{(2)}, T), \quad x_1^{(2)} = \varkappa(t_1^{(2)}, t_1^{(1)}, x_1^{(1)}, u_2, v_1')$$

Proceeding thus, after at most $n+m-1$ steps we obtain a unique pair of controls $u(a, b)$ and $v(a, b)$ on $[t_*, T)$.

For any number $\delta > 0$ we define the strategy

$$a(\delta) = \begin{Vmatrix} \varepsilon_1^t(\delta) \ldots \varepsilon_m^t(\delta) \\ \varphi_1^t(\delta) \ldots \varphi_m^t(\delta) \end{Vmatrix}$$

as follows:

$$\varepsilon_m^t(\delta) = T - t \tag{4.6}$$

$$W_{m-k+1}^-(t, x) \leqslant \sup_{\tau \in [t, T]} W_{m-k}^-(\tau, \varkappa(\tau, t, x, \varphi_k^t(\delta)(x), v)) + \frac{\delta}{2m}$$

$$\sup_{\tau \in [t, T]} W_{m-k}^-(\tau, \varkappa(\tau, t, x, u, v)) \leqslant$$

$$W_{m-k}^-(\varepsilon_k^t(\delta)(x, u, v), \varkappa(\varepsilon_k^t(\delta)(x, u, v), t, x, u, v)) + \frac{\delta}{2m}$$

$$\forall u \in D_1[t, T), \; v \in D_2[t, T), \; t \in [t_0, T], \; x \in X$$

$$k = 1, 2, \ldots, m-1$$

$$W_1^-(t, x) \leqslant H(\varkappa(T, t, \varphi_m^t(\delta)(x), v) + \frac{\delta}{m}, \quad \forall v \tag{4.7}$$

From (4.6), (4.7) follows the inequality

$$K(a(\delta), b, t_*, x_*) = H(\varkappa(T, t_*, x_*, a(\delta), b)) \geqslant W_{m*}^- - \delta \tag{4.8}$$

for all general positional piecewise-programmed stategies b of the second player. Analogously, a general positional piecewise-programmed strategy $b(\delta)$ of the second player exists such that

$$K(a, b(\delta), t_*, x_*) \leqslant W_{m_*}^+ + \delta \tag{4.9}$$

for all general positional piecewise-programmed strategies a. Inequality (4.5) follows from (4.8) and (4.9). Lemma 2 is proved in the same way.

Note. The strategy pair $a(\varepsilon/3), b(\varepsilon/3)$ forms an ε-equilibrium situation in the dynamic game $\Gamma(t_*, x_*)$ in the class of general positional piecewise-programmed strategies if only m is a sufficiently large number and the hypotheses of Theorem 1 are fulfilled.

5. From Theorem 1 follows:

Theorem 3. Let the dynamic system Σ satisfy condition 7) and function H be uniformly continuous. Then a function $V(t, x)$ such that

$$V(T, x) = H(x) \tag{5.1}$$

satisfying condition (5.2) or (5.3)

$$V(t, x) = \inf_{v \in D_2} \sup_{u \in D_1} \inf_{\tau \in [t, T]} V(\tau, \varkappa) = \sup_{u \in D_1} \inf_{v \in D_2} \sup_{\tau \in [t, T]} V(\tau, \varkappa) \tag{5.2}$$

$$V(t, x) = \inf_{\tau \in [t, T]} \inf_{v \in D_2} \sup_{u \in D_1} V(\tau, \varkappa) = \sup_{\tau \in [t, T]} \sup_{u \in D_1} \inf_{v \in D_2} V(\tau, \varkappa), \quad (\varkappa = \varkappa(\tau, t, x, u, v)) \tag{5.3}$$

for all $t_0 \leqslant t \leqslant T$, $x \in X$, is the value function of the dynamic game $\Gamma(t, x)$, $\operatorname{val}\Gamma(t, x)$ $V(t, x)$, $t_0 \leqslant t \leqslant T$, $x \in X$.

Proof. From (5.1) follows the inequality

$$V_1^-(t, x) = W_1^-(t, x) \leqslant V(t, x) \leqslant V_1^+(t, x) = W_1^+(t, x)$$

Then from (5.2) we obtain

$$W_n^-(t, x) \leqslant V(t, x) \leqslant W_n^+(t, x), \quad n = 2, 3, \ldots$$

or, if the stronger condition (5.3) is fulfilled, then

$$V_n^-(t, x) \leqslant V(t, x) \leqslant V_n^+(t, x), \quad n = 2, 3, \ldots$$

By Theorem 1 we have $V(t, x)$ $\operatorname{val}\Gamma(t, x)$.

From Theorem 2 follows

Theorem 4. Let the dynamic system Σ satisfy conditions 8) and 9) and function H be uniformly continuous. Then a function $V(t, x)$, satisfying condition (5.1) and such that condition (5.4) or (5.5)

$$V(t, x) = \inf_{v \in V} \sup_{u \in D_1} \inf_{\tau \in [t, T]} V(\tau, \varkappa) = \sup_{u \in U} \inf_{v \in D_2} \sup_{\tau \in [t, T]} V(\tau, \varkappa) \tag{5.4}$$

$$V(t, x) = \inf_{\tau \in [t, T]} \inf_{v \in V} \sup_{u \in D_1} V(\tau, \varkappa) = \sup_{\tau \in [t, T]} \sup_{u \in U} \inf_{v \in D_2} V(\tau, \varkappa) \tag{5.5}$$

$$(\varkappa = \varkappa(\tau, t, x, u, v))$$

is satisfied for all $t_0 \leqslant t \leqslant T$, $x \in X$, is the value function of dynamic game $\Gamma(t, x)$.

We note that (5.3) is a generalization of the equations in /8,9/. All Eqs.(5.2)—(5.5) can be used to find ε-optimal strategies in dynamic games $\Gamma(t, x)$ in the class of general positional piecewise-programmed strategies.

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